UNSTABLE STATES OF A WEIGHTLESS VISCOUS FLUID LAYER IN A ROTATING CYLINDRICAL VESSEL

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A layer of a viscous, weightless capillary fluid located on the lateral inside surface of a cylindrical vessel rotating at a constant angular velocity can be in a state of equilibrium relative to the cylinder – rotating together with the latter as a rigid body. If the contact angle at the ends of the cylindrical vessel is equal to $\pi/2$, then the layer may be in equilibrium with the free inside surface (having the form of a circular cylinder). Along with this trivial equilibrium state, it is also possible to obtain equilibrium configurations in which the free surface of the layer is symmetrical relative to the axis of rotation and is periodic in the direction of this axis. The goal of the present study is to examine such nontrivial axisymmetric states of relative equilibrium and analyze their stability.

Pukhnachev [1] examined the problem of branching of the state of rigid-body rotation of a circular cylindrical layer with a free internal and solid external surface. Critical Weber numbers corresponding to the branching of a state of rigid-body rotation with an axisymmetric periodic free surface were found in this investigation.

In our study, we examine all of the equilibrium configurations of a weightless rotating layer with an axisymmetric periodic free boundary. Here, we substantiate the branching conditions obtained in [1] [see Eq. (2.7)]. We also show that there exists a family of nontrivial axisymmetric states which do not branch from the circular cylindrical state.

Joseph and Preziosi [2] analytically and experimentally studied the problem of axisymmetric equilibrium configurations of a two-layer liquid filling a rotating vessel in the form of a cylindrical layer. In essence, Badratinova [3] studied the case when the liquid at the external boundary of the vessel is less dense than the internal fluid. Nontrivial axisymmetric modes of equilibrium of a rotating liquid column were examined in this investigation. Badratinova [3] reported critical values of parameters at which their branching could be expected to occur. Branching three-dimensional equilibrium configurations were observed in the experiments in [2]. The situation examined in the present study is another special case of a two-layer liquid in which the density of the internal liquid is zero or less than the density of the external liquid. In contrast to [2], we consider the case of a cylindrical vessel which is finite along the axis of rotation.

The studies [2, 4] validated a principle in accordance with which stable configurations of a two-layer liquid are associated with rigid-body motion minimizing a certain surface potential on the set of possible interfaces. However, the principle stated in [2] was not substantiated with sufficient rigor [the theorem which follows Eq. (2.39) in this study is erroneous]. The potential introduced in [2] is the "potential energy of the system in a rotating coordinate system." A similar potential was introduced earlier (see [5], p. 125) to study the stability of the equilibrium of rotating fluid. Validation of the principle of a potential-energy minimum for fluid-dynamics problems was begun in [6], where the author determined the stability of the equilibrium state of a liquid with a free boundary for cases in which this principle is valid. A similar determination of stability was made in [7, 8] for a viscous capillary liquid. The authors proved the validity of the Lagrange theorem: the equilibrium state of a liquid for which the second variation of potential energy is positive is stable. Studying the linear problem of the stability of the equilibrium of a viscous capillary liquid partially filling a vessel, the authors of [9] proved the validity of the inverse Lagrange theorem: if the second variation of potential energy can take negative values, then the equilibrium is unstable.

In the present investigation, stability is studied on the basis of an analogy with the Lagrange theorem and its inverse. In the study of stability in [2], the potential minimum

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was sought on a one-parameter family of equilibrium axisymmetric surfaces whose period depended on the characterizing parameter. Such a family is not allowable in a finite cylindrical vessel. The length of the latter should be a multiple of the wavelength of the disturbance. Thus, we cannot use the results obtained in [2] regarding the instability of nontrivial axisymmetric equilibrium configurations in an infinite cylindrical layer.

According to [2], the state of equilibrium with a cylindrical interface in an infinite vessel will always be unstable if the heavy liquid is located on the inside (is the rotating liquid column). When the heavy liquid is on the outside (is an infinite circular cylindrical layer), it is stable only at Weber numbers greater than four. The first statement is supported by the result obtained in [10], but the latter contradicts the well-known fact that an infinite layer is stable relative to random disturbances at Weber numbers greater than unity (this result follows from [10, 8] and [5, p. 165]). It should be noted that the equilibrium configurations observed in the experiments in [2] at Weber numbers less than four were not circular cylindrical. It may be that, of the two stable states, the state realized in an experiment is that which has the lower potential energy.

The behavior of a two-layer liquid in a gravitational field inside a rotating horizontal cylinder was studied experimentally in [11]. The authors examined two-layer systems in which the internal liquid was less dense than the external liquid. Experiments were also conducted for liquid partially filling a vessel and located on its lateral inside surface. As was shown in [2] and in our study, in the case of a weightless liquid layer, all of the equilibrium configurations are unstable in the presence of a noncylindrical free surface that does not intersect the axis of rotation. The situation is different for a liquid layer in a gravitational field. The experimental results in [11] indicate that secondary periodic (three-dimensional) flows are stable.

1. Equations for Equilibrium Modes. A circular cylindrical vessel of radius R and length L is rotated about its own axis at a constant angular velocity Ω. The vessel is partially filled with a weightless viscous fluid having a volume equal to $\pi(R^2 - R_0^2)L$. The contact angle is $\pi/2$. One possible state of the fluid is rigid-body rotation with the angular velocity Ω and a cylindrical free surface located the distance R_0 from the axis of rotation.

Examining a fluid in a state of equilibrium relative to a rotating cylinder, we pose the problem of finding all possible axisymmetric modes of the free surface that will not intersect the lateral surface of the cylinder and will project uniquely on this surface. Such modes are found from an analog of the Laplace equation which accounts for centrifugal forces in the balance of forces on the equilibrium free surface. We introduce dimensionless variables for our study, having chosen R_0, ΩR_0, $\rho\Omega^2 R_0^2$ as the scales of length, velocity, and pressure (ρ is the density of the fluid). Let η, α, and z be a cylindrical coordinate system rotating with the angular velocity Ω. The z axis of this system coincides with the axis of the cylinder. The ends of the cylindrical vessel are located in the planes z = 0, $z = \ell = L/R_0$. First we will examine the axisymmetric equilibrium modes for which the distance from the axis of rotation increases monotonically with motion along the free surface in the plane α = const from the end z = 0 to the end $z = \ell$. The equation of these modes can be represented in the form $z = Z(\eta)$, where the function $Z(\eta)$ satisfies the equilibrium equation [5]

$$H = -\frac{1}{2\eta}\left(\frac{\eta Z'}{(1+Z'^2)^{1/2}}\right)' = \frac{\beta}{2}\eta^2 - C. \tag{1.1}$$

Here and below, primes denote differentiation with respect to η; $\beta = \rho\Omega^2 R_0^3/\sigma$ is the Weber number (σ is surface tension); C is an unknown constant; H is the mean curvature of the surface $Z(\eta)$. The sign in front of the curvature is chosen with allowance for the above assumption that $Z'(\eta) > 0$.

The solution of differential equation (1.1) must satisfy the boundary conditions

$$Z'(\eta_0) = Z'(\eta_1) = \infty \tag{1.2}$$

and the integral relations

$$\int_{\eta_0}^{\eta_1} \eta^2 Z' d\eta = l, \quad \int_{\eta_0}^{\eta_1} Z' d\eta = l, \tag{1.3}$$

which, respectively, express that the contact angle is equal to $\pi/2$, the volume of the fluid is equal to $\pi(R^2 - R_0^2)L$, and the length of the cylindrical vessel is equal to L. In Eqs. (1.2) and (1.3), η_0 and η_1 are the largest and smallest values found for the distance of the equilibrium surface from the axis of rotation: $\eta_0 = \eta|_{z=0}$, $\eta_1 = \eta|_{z=\ell}$. For each solution of problem (1.1)-(1.3), the condition that the lateral surface of the cylinder not be intersected imposes the following limitation on the radius R:

$$R > \eta_1 R_0. \tag{1.4}$$

After integration, Eq. (1.1) gives

$$-\frac{\eta Z'}{(1+Z'^2)^{1/2}} = \frac{\beta}{8}\eta^4 - \frac{C}{2}\eta^2 - C_1 \tag{1.5}$$

(C_1 is an unknown constant).

The relative equilibrium modes for which the distance from the axis of rotation decreases monotonically with motion along a meridional arc from the bottom to the top end are determined by the equation $z = \ell - Z(\eta)$. We will refer to a relative equilibrium mode as simple if the free surface of the liquid projects uniquely on the planes of the ends of the vessel. Equations (1.2)-(1.5) establish a two-parameter (dependent on the parameters β, ℓ) family of simple axisymmetric equilibrium modes accurate to within the transformation $\hat{z} = \ell - z$.

Problem (1.2)-(1.5) is invariant relative to mirror reflection in the plane $z = \ell$. Thus, it can also be concluded that Eqs. (1.2)-(1.5) establish a two-parameter family of periodic axisymmetric modes of relative equilibrium. The dimensionless length of the vessel should be a multiple of the "half-period" ℓ with the contact angle $\pi/2$. If it is not a multiple of ℓ for the given β and ℓ, then the periodic solution will be the solution of the problem of equilibrium in the vessel for another contact angle.

2. Representation of the Solution in Formulas of Integration. Let us study a two-parameter family of simple equilibrium modes. We choose the following as independent parameters

$$\theta = \eta_0/\eta_1, \quad b = -\frac{\beta(1+\theta)\eta_1^3}{8}. \tag{2.1}$$

Passing to the limits at $\eta \to \eta_0$ and $\eta \to \eta_1$ in (1.5) and considering conditions (1.2), we obtain expressions for the constants C and C_1 in terms of the parameters θ, b, and η_1:

$$C = 2[1 - b(1+\theta^2)]/[\eta_1(1+\theta)], \quad C_1 = \theta\eta_1(1+b\theta)/(1+\theta). \tag{2.2}$$

We now change over to the parameters b, θ and the new variables $x = z/\eta_1$, $r = \eta/\eta_1$ in problem (1.2)-(1.5). After integration, we find from (1.5) that

$$x = X(r, \theta, b) = \int_\theta^r \frac{u(\tau, \theta, b)\, d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}, \tag{2.3}$$

where u represents the function

$$u(r, \theta, b) = \frac{r^2 + \theta - b(1-r^2)(r^2-\theta^2)}{\sqrt{(1+r)(r+\theta)[1 + 2b(r^2+\theta) - b^2(1-r^2)(r^2-\theta^2)]}}. \tag{2.4}$$

We use conditions (1.3) to find the dependence of η_1 and ℓ on θ and b:

$$\eta_1 = \left[\int_\theta^1 \frac{u(\tau, \theta, b)\, d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}\right]^{1/2} \left[\int_\theta^1 \frac{\tau^2 u(\tau, \theta, b)\, d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}\right]^{-1/2}; \tag{2.5}$$

$$l = F(\theta, b) = \left[\int_\theta^1 \frac{u(\tau, \theta, b)\, d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}\right]^{3/2} \left[\int_\theta^1 \frac{\tau^2 u(\tau, \theta, b)\, d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}\right]^{-1/2}. \tag{2.6}$$

The value of θ belongs to the interval (0, 1) for each equilibrium mode. If $\theta = 0$, then the boundary condition on the end z = 0 is violated $\left(\lim\limits_{r=\theta\to 0} \frac{dX}{dr} = 0\right)$. The equilibrium surface touches the surface of this end. In a special case ($b = \theta = 0$), it has the form of a hemisphere of the radius $\ell = \sqrt{1.5}$. At $\theta \to 1$, the equilibrium modes approach a circular cylindrical surface. Let us now change over to the new variable of integration t in Eqs. (2.5)-(2.6) by means of the formula $\tau = (1+\theta)/2 + (1-\theta)t/2$ and let us set $\theta = \theta^* = 1$ in these equations. We obtain $\eta_1 \equiv \eta_1^* = 1$, $\ell = \pi/\sqrt{1+4b}$. From this, with allowance for (2.1), we obtain the formula

$$\beta^* = -4b^* = 1 - (\pi/l)^2, \tag{2.7}$$

coinciding with the condition found in [1] for the branching of an equilibrium state with a circular cylindrical surface.

As regards its physical meaning, the parameter $\beta > 0$. We can thus conclude the following from (2.7). Simple axisymmetric modes of relative equilibrium branching from the circular cylindrical mode exist only at $\ell > \pi$. Figure 1 shows the dependence of relative length ℓ on the parameter θ at certain values of b. It is apparent that simple axisymmetric equilibrium surfaces also exist at $\sqrt{1.5} < \ell < \pi$ but do not branch from the circular cylindrical surface. With fixed $\ell \in (\sqrt{1.5}, \pi)$, such axisymmetric solutions exist beginning with the value $\theta = 0$ (at which the equilibrium surface intersects the rotation axis) and ending with the value θ_0 (corresponding to $b = 0$). At $b = 0$, $\beta = 0$, i.e., the cylindrical vessel is at rest.

It can be shown that the radicand in (2.4) is positive at all $r \in [\theta, 1]$ only if the following inequality is satisfied

$$b > -0{,}5(1+\theta)^{-1}. \tag{2.8}$$

At $b = -0.5(1 + \theta)^{-1}$, the expression in square brackets in (2.4) is equal to $(1 - r^2)(4 + 4\theta - r^2 + \theta^2)/4(1 + \theta)^2$. The integrand in (2.3) is thus proportional to $(1 - \tau)^{-1}$. At $r \to 1$, integral (2.3) diverges and the "half-period" $\ell \to \infty$.

It follows from (2.8) that the minimum value of b at which simple equilibrium modes exist is attained at $\theta = 0$ and is equal to -0.5. Thus, a "degenerate" solution for which contacting gas bubbles are formed in the fluid in a state of relative equilibrium exists up to the value $\ell = \infty$. As can be seen from Fig. 1, such solutions do not exist when $\ell < \sqrt{1.5}$.

Figure 2 shows the region D corresponding to the existence of simple axisymmetric equilibrium modes in the space of the initial parameters ℓ, β. Curves $\ell_0(\beta)$ and $\ell = \ell_1(\beta) = \pi/(1 - \beta)$ correspond to $\theta = 0$ and 1. Curve $\ell_0(\beta)$ has the vertical asymptote $\beta = 4$. The region D is enclosed by curves $\ell_0(\beta)$, $\ell_1(\beta)$. It is evident from Fig. 2 that nontrivial modes exist at $\beta \in (0, 4)$.

Let a simple axisymmetric mode be characterized by the parameters β and ℓ, for which the point $(\beta, \ell) \in D$. For this mode to be the solution of the problem of equilibrium in a cylinder of length ℓR_0 and radius R, inequality (1.4) must be satisfied. Numerical calculations showed that (1.4) is satisfied throughout the region D at $R > R_* \approx 1.36R_0$. In the next paragraph, we make the assumption that the radius of the cylinder R satisfies inequality (1.4). We thus analyze the stability of all modes from region D.

We will prove the following hypothesis. There are no periodic solutions to the problem of relative equilibrium in a cylindrical vessel for which a section of the free surface corresponding to one half-period projects uniquely on the plane z = const. Let the angular velocity Ω (or $\beta = \rho\Omega^2R_0^3/\sigma$) and the degree of filling R_0/R of the cylinder be given. If such a solution exists, then the half-period can contain two simple sections which, while being continuations of one another, are not mirror reflections of each other across the orthogonal axis of the z plane. Each section is the solution of the problem of simple equilibrium modes with the angular velocity Ω. However, generally speaking, with the "characteristic" degree of filling R_i/R and the "characteristic" value of the Weber number $\beta_i = \rho\Omega^2R_i^3/\sigma$, i = 1, 2.

Each section is uniquely characterized by its own set of parameters θ_i, b_i [$\theta_i = \eta_{0i}/\eta_{1i}$, $b_i = -\beta_i(1 + \theta_i)\eta_{1i}^3/8$]. When expressed in the initial physical variables, curvature and the distance of the free surface from the axis of rotation should be continuous with passage through a contact point. The distance of the free surface from the rotation axis will be minimal or maximal at such points. In the first case, the below inequality follows from the continuity of the distance

$$\eta_{11}R_1 = \eta_{12}R_2, \tag{2.9}$$

while in the second case

$$\theta_1\eta_{11}R_1 = \theta_2\eta_{12}R_2. \tag{2.10}$$

With allowance for these equalities, we can use Eqs. (1.1) and (2.2) to obtain the following curvature continuity conditions. For the first case

$$\frac{1 - b_1(1 - \theta_1^2)}{1 - b_2(1 - \theta_2^2)^2} = \frac{(1+\theta_1)}{(1+\theta_2)}, \tag{2.11}$$

while for the second case

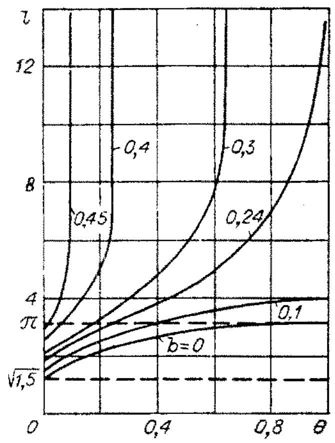

Fig. 1

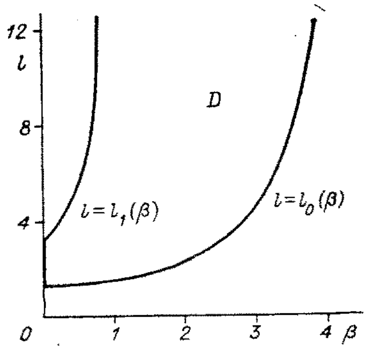

Fig. 2

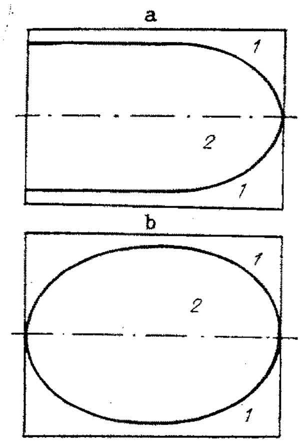

Fig. 3

$$\frac{1-b_1\left(1-\theta_1^2\right)}{1-b_2\left(1-\theta_2^2\right)}=\frac{(1+\theta_1)\,\theta_2}{(1+\theta_2)\,\theta_1}. \tag{2.12}$$

With allowance for (2.9), (2.10), we use the definition of the parameters b_i, β_i, and β to obtain relations between the parameters b_i and θ_i in the first case

$$b_2=b_1(1+\theta_2)/(1+\theta_1), \tag{2.13}$$

while in the second case

$$b_2=b_1(1+\theta_2)\,\theta_1^3/(1+\theta_1)\,\theta_2^3. \tag{2.14}$$

The given sections of the free surface can be solutions of the problem of simple equilibrium modes if the following inequality is satisfied simultaneously [see condition (2.8)]

$$b_i>-0{,}5(1+\theta_i)^{-1},\quad i=1,\ 2. \tag{2.15}$$

We express the parameters b_i through θ_i from Eqs. (2.11) and (2.13) [(2.12) and (2.14), respectively] and we write conditions (2.15) for them. As a result, in each case we obtain a system of inequalities relative to the parameters θ_1 and θ_2, which are inconsistent when θ_1, $\theta_2 \in [0, 1]$. This proves the hypothesis.

3. Instability of Axisymmetric Periodic Equilibrium Configurations. On the basis of the analog of the Lagrange theorem and its inverse, the problem of the stability of relative equilibrium of a fluid in a vessel rotating at a constant angular velocity reduces [5] to determination of the sign of the second variation of the functional $U = \sigma|\Gamma| + \tilde{\sigma}|\Sigma| + \sigma_0|\Sigma_0| - \Omega^2 I/2$, where $|\Gamma|$, $|\Sigma|$, $|\Sigma_0|$ are the area of the free surface and the areas of the surfaces of contact of the liquid and gas with the boundary of the vessel; $\tilde{\sigma}$ and σ_0 represent the values for surface tension on Σ, Σ_0; $I=\rho\int\limits_V r^2\,dV$ (r is the distance from the axis of rotation) is the moment of inertia of the liquid.

Let Γ be a simple equilibrium mode characterized by the parameters θ and b, and let $N(S, \alpha)$ be the normal component of a perturbation of the free boundary referred to η_1. The problem of the stability of the surface Γ reduces [5] to determination of the eigenvalues of the following linear boundary-value problem relative to $N(S, \alpha)$:

$$-\frac{1}{r}\frac{\partial}{\partial S}\left(r\frac{\partial N}{\partial S}\right)-\frac{1}{r^2}\frac{\partial^2 N}{\partial\alpha^2}+aN+\mu=\lambda N; \tag{3.1}$$

$$\frac{\partial N}{\partial S}=0\quad(S=0),\quad \frac{\partial N}{\partial S}=0\quad(S=S_1); \tag{3.2}$$

$$\int\limits_0^{S_1}\int\limits_0^{2\pi} Nr\,dS\,d\alpha=0\ \begin{pmatrix}0\leqslant S\leqslant S_1\\ 0\leqslant\alpha\leqslant 2\pi\end{pmatrix}. \tag{3.3}$$

Here,

$$S=(1+\theta)\int\limits_\theta^r\frac{\tau G(\tau,\,\theta,\,b)\,d\tau}{\sqrt{(1-\tau^2)(\tau^2-\theta^2)}} \tag{3.4}$$

is the ratio of the length of an arc Γ, reckoned from the end $z = 0$, to the parameter η_1; S_1 is the value of S at $r = 1$;

$$G(\tau,\ \theta,\ b)=1/\sqrt{1+2b(\tau^2+\theta)-b^2(1-\tau^2)(\tau^2-\theta^2)}. \tag{3.5}$$

The function a is determined from the formula $\eta_1^2 a = -\partial H/\partial n - 4H^2 + 2K$ [$K = Z'Z''(1 + Z'^2)^{-2}$ is the Gaussian curvature Γ, $\partial/\partial n = \mathbf{n}\cdot\nabla$, $\mathbf{n}$ is a unit normal to Γ, directed inside the region occupied by the gas] and is expressed in terms of the parameters θ and b in the form

$$a=-2[4bP+(P/r^2-Q)^2+Q^2](1+\theta)^{-2},$$
$$P=r^2+\theta-b(1-r^2)(r^2-\theta^2),\quad Q=1-b(1-2r^2+\theta^2). \tag{3.6}$$

Conditions (3.2) follow from the assumption that, with disturbance of the equilibrium state, the dynamic contact angle at the ends is equal to $\pi/2$ – the static contact angle. Equation (3.3) expresses the condition of conservation of fluid volume.

The eigenvalues of problem (3.1)-(3.6) are real. If the lowest eigenvalue λ_* is positive, then the corresponding equilibrium configuration is stable. If λ_* is negative, the corresponding equilibrium configuration is unstable. If the function $N(S, \alpha)$ is represented in the form of a series

$$N=\sum_{m=0}^{\infty}[\varphi_m(S)\cos m\alpha+\psi_m(S)\sin m\alpha],$$

then it can be shown [5] that $\lambda_*=\min\limits_{m=0,1}(\lambda_m)$, where λ_1 is the smallest eigenvalue of the problem

$$\frac{1}{r}\frac{d}{dS}\left(r\frac{d\varphi_m}{dS}\right)-\left(a+\frac{m^2}{r^2}\right)\varphi_m+\lambda\varphi_m=0\quad(0\leqslant S\leqslant S_1); \tag{3.7}$$

$$\frac{d\varphi_m}{dS}=0,\quad S=0,S_1 \tag{3.8}$$

when m = 1, while λ_0 is the smallest eigenvalue of problem (3.7) when m = 0. Here, we also have the condition

$$\int_0^{S_1}\varphi_0 r\,dS=0. \tag{3.9}$$

Problems (3.7)-(3.9) were solved by the Galerkin–Ritz method. We used the complete system of eigenfunctions $y_k(S)$ introduced in [3]. At m = 0 $y_k = Y_k - I_k/I_0$ (k = 1, 2,...). Here,

$$Y_1=2S^3-3S^2S_1,\quad Y_k=S^2(S-S_1)^k,\quad I_0=\int_0^{S_1}r\,dS,\quad I_k=\int_0^{S_1}Y_k r\,dS.$$

At m = 1, we assume that $y_k = Y_k$. The method of solution was described in [3]. The difference between our approach here and [3] is that we allow perturbations leading to displacement of the center of mass of the liquid layer from the axis of rotation. Numerical calculations were performed for simple equilibrium modes with the parameter $\theta \in [0, 1)$, $b \in (-0.5 \times (1 + \theta)^{-1}, 0]$. It was found for all of the equilibrium modes that the eigenvalue λ_0 is positive (and equal to 0 only at $\ell > \pi$, $\theta = 1$, i.e., at branch points) while λ_1 is negative. This proves the following theorem.

In a rotating cylindrical vessel with a contact angle equal to $\pi/2$ at the ends, all possible axisymmetric equilibrium configurations of the layer are unstable when the distance from the axis of rotation changes monotonically with motion along a meridional arc from one end to the other. Axisymmetric disturbances are dangerous.

It follows from this that the rigid-body rotation of a liquid layer with an axisymmetric periodic free boundary located on the inside lateral surface of a cylinder of infinite length is unstable. This conclusion agrees with the result in [2]. In contrast to [2], however, the perturbations studied here had the same periods along the z axis as did the equilibrium mode that was analyzed for stability. This is evident from condition (3.8), the points S = 0, S_1 corresponding to z = 0, z = ℓ. The validity of the next theorem follows from this and the fact that the size of each half-period at whose ends conditions (1.2) are satisfied remains the same in the case of dangerous axisymmetric perturbations.

In a rotating cylindrical vessel of finite length with a contact angle of $\pi/2$ at the ends, all possible noncylindrical axisymmetric states of relative equilibrium of a viscous liquid layer on the inside lateral surface are unstable. In accordance with this theorem, when the contact angle is $\pi/2$, all equilibrium configurations with a periodic free surface and a free surface consisting of an odd number of half-periods on whose ends conditions (1.2) are satisfied are unstable. Here, the inverse of the Lagrange theorem [9] guarantees an increase in the mean-square perturbation $N(S, \alpha)$ during each half-period. If the equilibrium surface contains even one half-period that does not contact the ends, then the model chosen to describe the behavior of perturbations near the ends is unimportant insofar as deciding whether or not this surface will be unstable: either the adhesion model or the slip model can be used, and the dependence of the dynamic angle on the velocity of the three-phase contact line can be accounted for or ignored.

The following more general theorem is also valid. As regards the case of a rotating cylindrical vessel of relative length ℓ, if the free surface for the state of rigid-body rotation of a meridional section of a fluid layer $\eta(z)$ contains two stationary points z_0 and z_1 for which $\eta'(z_0) = \eta'(z_1) = 0$, $0 \neq z_0 < z_1 \neq \ell$, then this state will be unstable.

This problem follows from the fact that, within the class of axisymmetric perturbations, there are perturbations $\varphi_0(S)$, for which the volume of the liquid within the region $z_0 \leqslant z \leqslant z_1$, remains constant and condition (3.8) is satisfied at the values $S = 0$, S_1 corresponding to points z_0, z_1.

When the contact angles at the ends are not equal to $\pi/2$, a state of rigid-body rotation of a layer with a free boundary that is periodic along the axis of rotation can exist in a rotating cylindrical vessel. For this state to exist, the length of the cylinder must be a multiple of the period and the sum of the contact angles at the ends must be equal to π. Such a state is unstable, since the meridional section of each period contains two stationary points.

We should emphasize that we did not examine the stability of equilibrium configurations of layers in cases in which the complete meridional arc of the free surface contains either no stationary points or only one such point between the ends of the rotating cylinder. Such equilibrium configurations can be realized at low Weber numbers in a sufficiently short cylindrical vessel. Some of these configurations can be expected to be stable.

Stability can be determined analytically [8] (also see [5], p. 162) for the rigid-body rotation of a cylindrical viscous layer in a cylinder of finite length. Among the possible perturbations are those for which the value of the functional U remains constant. Regardless of the Weber number, U remains constant for perturbations that do not cause the free surface to deviate from the cylindrical form but do displace it from the rotation axis. This indicates that, assuming that the center of mass of the layer may be displaced from this axis, the equilibrium may be only neutrally stable. If we ignore perturbations which displace the center of mass, then the equilibrium state will be unstable (relative to axisymmetric perturbations) at $\beta < \beta_* = (1 - \pi^2/\ell^2)$ and stable at $\beta > \beta_*$. As was noted in Part 2, the cylindrical state branches at $\beta = \beta_*$. The value of β_* is equal to unity at $\ell \to \infty$.

According to the results of our study, simple equilibrium surfaces are unstable at $\theta = 0$. Also unstable are the configurations depicted schematically in Fig. 3a and b (1 and 2 denote the regions occupied by the liquid and the gas). This conclusion is consistent with the conclusion made in [2] that equilibrium configurations which intersect the rotation axis in an infinite cylinder are stable, since perturbations satisfying different boundary conditions were analyzed in the given case. Contact of a gas bubble with one end of the cylinder at just one point is all that is necessary for instability to occur.

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DEVELOPMENT OF SEPARATION IN THE REGION WHERE A SHOCK INTERACTS WITH A TURBULENT BOUNDARY LAYER PERTURBED BY RAREFACTION WAVES

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The solution of many supersonic gas dynamic problems requires studying how the turbulent boundary layer interacts with various perturbations, such as shocks and expansion waves. The essential feature of such flows, compared to single interactions, is that upstream perturbations can cause downstream relaxation effects in the boundary layer. Under such conditions, relaxation properties of the flow, in particular its separation resistance, can depend on the distance from the perturbation, and also on its type and intensity.

Current research indicates that the separation properties of a turbulent boundary layer in various situations depend significantly on how well its average velocity profile is filled out. The behavior is characteristic for the development of a turbulent boundary layer on a plate [1-3]. As the Reynolds number increases to $Re_\delta \approx 10^5$, the velocity profile is observed to be less filled out and hence less able to resist separation; while the velocity profile becomes more filled out in developed flow and at higher Reynolds numbers, and hence it becomes more resistant to separation. Analogous features are noted for perturbed boundary layers in which various perturbations fill out the velocity profile. For example, theoretical analysis led to the conclusion that separation could be suppressed in accelerated flows [4]. The increase in the critical intensity of a shock in flows with boundary-layer suction and discharge has been recorded experimentally [5]. An analogous effect was observed in a turbulent boundary layer which had been perturbed by a shock [6]. According to [5], such an effect of filling out the velocity profile is explained by the total pressure growth in a characteristic wall region of the boundary layer, where its separation properties are determined. At the same time, it is obvious that in general the change in the separation properties of perturbed boundary layers can be related not only to the transformation of the average velocity profile but also to a change in the effective viscosity in the wall region. Undoubtedly the role of each of these factors should be further refined. The results of the aforementioned studies create interest in the experimental study of how a shock interacts with a turbulent boundary layer which has been perturbed by expansion waves. This situation is an extreme one for testing the use of various turbulence models in current numerical calculations.

This paper is the result of an experimental and numerical study of how a turbulent boundary layer interacts with expansion waves and a shock as it flows over wedges. The study is a combined effort of the Institute of Theoretical and Applied Mechanics, Siberian Branch, Russian Academy of Sciences (ITPM SO RAN) and NASA Ames (USA). The experiments were done at ITPM SO RAN under adiabatic surface conditions in the T-313 and T-325 hypersonic wind tunnels (cross sections of 0.6 × 0.6 m and 0.2 × 0.2 m in the working section). The models were flat wedges of fixed height h = 15 mm (for T-313) and 6 mm (for T-325) with leeward inclination angles of β = 8°, 25°, and 45°. The relatively large width of the models (b/h = 20-26.7) and the use of boundary plates eliminated three-dimensional edge effects on the flow characteristics near the plane of symmetry. Turbulent flow was developed on the horizontal surface

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